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Berry-Esseen Bounds of Approximate Bayes Estimators for the Discretely Observed Ornstein-Uhlenbeck Process

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ABSTRACT
Using random, nonrandom and mixed normings, the paper obtains uniform rates of weak convergence to the standard normal distribution of the distribution of several approximate Bayes estimators and approximate maximum a posteriori estimators of the drift parameter in the discretely observed Ornstein-Uhlenbeck process from high frequency data.

KEYWORDS
Approximate Bayes estimator; Approximate maximum a posteriori estimator; Bayesian computation; Discrete observations; High frequency data; Itô stochastic differential equation; Monte Carlo method; Ornstein-Uhlenbeck process; Rate of weak convergence; Uniform Berry-Esseen type bound.

1. Introduction

The Ornstein-Uhlenbeck process, also known as the Vasicek model in finance, is widely used in modeling short term interest rates and bond pricing. Many nonlinear diffusion processes can be represented as a functional of Ornstein-Uhlenbeck process via state space transforms. Also this process is the building block in stochastic volatility modelling where generalization to a (fractional) Levy process innovation, which can incorporate heavy tailed distributions and long memory effect by superposition, see, e.g, [1]. Though the models are continuous, in practice, for example in algorithmic trading, the data are discrete and of possible high frequency nature. In view of this, it becomes necessary to estimate the unknown parameters in the model from discrete high frequency data. In this paper, we assume constant volatility and without loss of generality assume it to be one. To estimate the drift parameter, we adopt approximate Bayes estimation method and approximate maximum a posteriori method. We study the accuracy of the distributional approximation of the estimators from high frequency data.

We are motivated by the advent of complete record of quotes or transaction prices for many financial assets. Although market microstructure effects (e.g., discreteness

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of prices, bid/ask bounce, irregular trading, etc.) means that there is a mismatch between asset pricing theory based on semimartingales and the data at every time intervals, it does suggest the desirability of establishing an asymptotic distribution theory of estimation as we use more and more high frequency observations where we have returns over increasingly finer time points. We concentrate on refined central limit theorem, i.e., Berry-Esseen theorem. See the monograph [2] for recent results on approximate likelihood asymptotics and approximate Bayes asymptotics for drift estimation of discretely observed diffusions based on high frequency data.

The paper deals with computational Bayesian method for parameter estimation. In computational Bayesian method, one seeks approximation of the posterior distribution. [3] studied approximate Bayes estimation for discrete stochastic processes which include birth and death process and branching process, where they used normal approximation of posterior, which is implied from the Bernstein-von Mises phenomenon. Our approximate Bayesian estimation method for diffusions uses Riemann type approximation of posterior distribution. An alternative way is to use particle filtering where the approximation of posterior distribution is done by sequential monte-carlo method. More precisely, in the particle filtering method, which is a sequential Bayes method, the posterior distribution is approximated by a large set of Dirac-delta masses (samples or particles) that evolve randomly in time according to the dynamics of the model and observations. Since particles are interacting, classical i.i.d. limit results are not applicable. We will study particle asymptotics in a future paper. See [4] for this exciting area of particle filtering.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis on which is defined the Ornstein-Uhlenbeck process $\{X_t\}$ satisfying the Itô stochastic differential equation

$$dX_t = \theta\ X_t\ dt\ +\ dW_t,\ t \geq 0, \quad X_0 = 0 \tag{1.1}$$

where $\{W_t\}$ is a standard Wiener process with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta < 0$ is the unknown parameter to be estimated on the basis of discrete observations of the process $\{X_t\}$ at times $0 = t_0 < t_1 < \cdots t_n = T$ with $t_i - t_{i-1} = \frac{T}{n}, i = 1, 2 \cdots, n$. Denote $X_0^{T,n} = \{X_{t_0}, X_{t_1}, \cdots, X_{t_n}\}$. We assume equally spaced data. Note that inclusion of the extreme two points of observations is very important. If one excludes the extreme two observations, one can not take the advantage of sufficiency and Rao-Blackwellization. We assume two types of increasingly high frequency data with long observation time: (1) $T \to \infty, n \to \infty, \frac{T}{\sqrt{n}} \to 0$; (2) $T \to \infty, n \to \infty, \frac{T}{n^{2/3}} \to 0$. At the end we consider asymptotics with $n \to \infty, \quad T$ fixed.

Let the continuous realization $\{X_t, 0 \leq t \leq T\}$ be denoted by X_0^T. Let P_θ^T be the measure generated on the space (C_T, B_T) of continuous functions on $[0, T]$ with the associated Borel σ-algebra B_T under the supremum norm by the process X_0^T and let P_0^T be the standard Wiener measure. It is well known by Girsanov theorem that when θ is the true value of the parameter P_θ^T is absolutely continuous with respect to P_0^T and the Radon-Nikodym derivative (conditional likelihood) of P_θ^T with respect to P_0^T based on X_0^T is given by

$$l_T(\theta) := \frac{dP_\theta^T}{dP_0^T}(X_0^T) = \exp\left\{\theta \int_0^T X_t dX_t - \frac{\theta^2}{2}\int_0^T X_t^2 dt\right\}. \tag{1.2}$$

Consider the score function, the derivative of the conditional log-likelihood function, which is given by

$$L_{T,1}(\theta) := \int_0^T X_t dX_t - \theta \int_0^T X_t^2 dt. \tag{1.3}$$

A solution of the conditional likelihood equation $l_{T,1}(\theta) = 0$ provides the (conditional) maximum likelihood estimate (MLE)

$$\theta_{T,1} := \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}. \tag{1.4}$$

Strictly speaking, $\theta_{T,1}$ is not the maximum likelihood estimate of θ since $\theta_{T,1}$ may take positive values where as the parameter θ is assumed to be strictly negative. Nevertheless, this definition of MLE is widely used. It is well known that $\theta_{T,1}$ is strongly consistent and $T^{1/2}(\theta_{T,1} - \theta)$ asymptotically $\mathcal{N}(0, -2\theta)$ distributed as $T \to \infty$ (see [5], [2]). [6] obtained weak convergence bound of the order $O(T^{-1/2})$ for the MLE.

In the stationary case where X_0 has $N(0, -2\theta)$ distribution, the log-likelihood function is given by

$$\check{L}_T(\theta) := \frac{1}{2}\log\left(-\frac{\theta}{\pi}\right) - \frac{\theta^2}{2}\int_0^T X_t^2 dt + \left(\frac{\theta}{2}\right)[X_T^2 + X_0^2] + \frac{\theta T}{2}.$$

In this case, the likelihood equation has two solutions:

$$\check{\theta}_{T,1} = \frac{[X_T^2 + X_0^2 - T] + \{[X_T^2 + X_0^2]^2 - \pi \int_0^T X_t^2 dt\}^{1/2}}{4\int_0^T X_t^2 dt},$$

$$\check{\theta}_{T,2} = \frac{[X_T^2 + X_0^2 - T] - \{[X_T^2 + X_0^2]^2 - \pi \int_0^T X_t^2 dt\}^{1/2}}{4\int_0^T X_t^2 dt}.$$

In this paper we will study the model (1.1). Using Itô formula, the score function $L_{T,1}(\theta)$ can be written as

$$L_{T,2}(\theta) := \frac{1}{2}X_T^2 - \int_0^T (\theta X_t^2 + \frac{1}{2})dt. \tag{1.5}$$

$$\theta_{T,2} := \frac{X_T^2 - T}{2\int_0^T X_t^2 dt}. \tag{1.6}$$

Consider the contrast function

$$L_{T,3}(\theta) := -\int_0^T (\theta X_t^2 + \frac{1}{2})dt \tag{1.7}$$

and the minimum contrast estimate (MCE)

$$\theta_{T,3} := -\frac{T}{2\int_0^T X_t^2 dt.} \tag{1.8}$$

The M-estimator is reduced to the minimum contrast estimator. It is well known that $\theta_{T,3}$ is strongly consistent and asymptotically $\mathcal{N}(0, -2\theta)$ distributed as $T \to \infty$ (see [7], [8]). The large deviations of $\theta_{T,1}$ and $\theta_{T,3}$ were obtained in [9]. [10] showed that $\theta_{T,3}$ has the Berry-Esseen bound of the order $O(T^{-1/2})$.

Let us introduce the Bayes procedure. Let $\lambda(\cdot)$ be the prior density which is assumed to be continuous and positive on Θ. Let $l(\theta - \phi)$ be a smooth loss function. The posterior density is given by

$$\Lambda_{T,i}(\theta|X_0^T) := \frac{\lambda(\theta)L_{T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{T,i}(\theta)d\theta}, \quad i = 1, 2, 3.$$

The Bayes risk is defined as the expectation of the loss function where the expecation is taken over the probability distribution of θ. A Bayes estimator which minimizes the Bayes risk is defined as

$$\tilde{\theta}_{T,i} := \arg\inf_{\theta\in\Theta} \int_\Theta l(\theta, \phi)\Lambda_{T,i}(\phi|X_0^T)d\phi, \quad i = 1, 2, 3.$$

For squared error loss function $l(\theta, \phi) = (\theta - \phi)^2$, the Bayes estimator is the posterior mean, which is given by

$$\tilde{\theta}_{T,i} = \int_\Theta \theta\Lambda_{T,i}(\theta|X_0^T)d\theta, \quad i = 1, 2, 3.$$

With a linear loss function $l(\theta, \phi) = a|\theta - \phi|$ with $a > 0$, the Bayes estimator is the posterior median which is useful in robust statistics. With a linear loss function, which assigns different weights $a, b > 0$ to over or sub estimation, $l(\theta, \phi) = a|\theta - \phi|$ for $\theta - \phi \geq 0$, and $l(\theta, \phi) = b|\theta - \phi|$ for $\theta - \phi < 0$, the Bayes estimator is the posterior quantile. The maximum a posteriori (MAP) estimator, which is the posterior mode, is given by

$$\hat{\theta}_{T,i} := \arg\max \Lambda_{T,i}(\theta|X_0^T), \quad i = 1, 2, 3.$$

The maximum probability estimator (MPE) in the sense of [11] (see also [12]) is defined as

$$\bar{\theta}_{T,i} = \arg\max_{\theta\in\Theta} \int_{\theta - T^{-1/2}}^{\theta + T^{-1/2}} L_{T,i}(X_0^T|\phi)d\phi, \quad i = 1, 2, 3.$$

It is known that MPE is asymptotically equivalent to the Bayes estimator with uniform prior distribution on a small interval and efficient. As it is a regular case, the Bernstein-von Mises theorem applies and the posterior is asymptotically normal (see [13]). [13] obtained weak convergence bound of the order $O(T^{-1/2})$ for the posterior distribution and Bayes estimator using nonrandom norming. [14] obtained weak convergence bound of the order $O(T^{-1/2})$ for the MLE and Bayes estimators using two different random normings which are useful for computation of a confidence interval.

Continuous observation is impossible to obtain in practice. Based one discrete observations of X_t at the equispaced time points $0 = t_0 < t_1 < \ldots < t_n = T$ with $t_j - t_{j-1} = T/n, j = 1, 2, \ldots, n$, we consider the following three contrast functions which are discretizations of $L_{T,1},\ L_{T,2},\ L_{T,3}$ respectively:

$$\begin{aligned} L_{n,T,1}(\theta) &:= \theta \sum_{j=1}^{n} X_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) - \frac{\theta^2}{2} \sum_{j=1}^{n} X_{t_{j-1}}^2 (t_j - t_{j-1}), \\ L_{n,T,2}(\theta) &:= \frac{\theta}{2}(X_T^2 - T) - \frac{\theta^2}{2} \sum_{j=1}^{n} X_{t_{j-1}}^2 (t_j - t_{j-1}), \\ L_{n,T,3}(\theta) &:= -\frac{\theta^2}{2} \sum_{j=1}^{n} X_{t_{j-1}}^2 (t_j - t_{j-1}). \end{aligned}$$

Here the first one is based on Itô approximation of the stochastic integral and Euler approximation of the ordinary integral (this also produces the conditional least squares estimator), the second one is based on Stratonovich approximation of the stochastic integral and Euler approximation of the ordinary integral, and the third one is based Euler approximation of the ordinary integral. Define the estimators

$$\theta_{n,T,i} := \arg\max_{\theta} L_{n,T,i}, \quad i = 1, 2, 3$$

which are given by

$$\theta_{n,T,1} = \frac{\sum_{j=1}^{n} X_{t_{j-1}}(X_{t_j} - X_{t_{j-1}})}{\sum_{i=1}^{n} X_{t_{j-1}}^2 (t_j - t_{j-1})},$$

$$\theta_{n,T,2} = \frac{X_T^2 - T}{2\sum_{j=1}^{n} X_{t_{j-1}}^2 (t_j - t_{j-1})},$$

$$\theta_{n,T,3} = \frac{-T}{2\sum_{i=1}^{n} X_{t_{j-1}}^2 (t_j - t_{j-1})}.$$

[15] obtained weak convergence bounds for the approximate maximum likelihood estimators (AMLEs) $\theta_{n,T,1}$ and $\theta_{n,T,2}$ where the later one is shown to have sharper bound than the former. The weak convergence bound for the approximate minimum contrast estimator (AMCE) $\theta_{n,T,3}$ was obtained in [16] where it is shown that it has the same error bound as the estimator $\theta_{n,T,2}$, and moreover this estimator is robust, asymptotically efficient and easy to simulate. Approximate maximum probability estimators (AMPEs) are defined as

$$\bar{\theta}_{n,T,i} = \arg\max_{\theta \in \Theta} \int_{\theta - T^{-1/2}}^{\theta + T^{-1/2}} L_{n,T,i}(X_0^{n,T} | \phi) d\phi, \quad i = 1, 2, 3.$$

The bound in probability on the difference between the AMLE and the continuous MLE was obtained by [17] for a more general nonlinear diffusion. For the O-U process, similar bound on the difference between AMCE and the continuous MCE was obtained in [16].

First order asymptotic theory of approximate Bayes estimators (ABEs) for discretely observed nonlinear diffusion processes was first introduced and studied in [2]. In this paper, beyond the first order asymptotic theory, we obtain the Berry-Esseen bounds of the ABEs for the Ornstein-Uhlenbeck process.

We also study approximate maximum a posteriori estimators (AMAPEs). The AML estimate can be seen as an approximate Bayes estimate when the loss function is not specified. The AMAP estimate incorporates prior information. The difference between AMAP and AML estimation lies in the assumption of an appropriate prior distribution of the parameters to be estimated.

The approximate posterior densities are defined as

$$\Lambda_{n,T,i}(\theta|X_0^{n,T}) := \frac{\lambda(\theta)L_{n,T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{n,T,i}(\theta)d\theta}, \qquad i = 1, 2, 3.$$

We consider squared error loss function. The approximate Bayes estimators (ABEs), which are the approximate posterior means, are given by

$$\tilde{\theta}_{n,T,i} := \int_\Theta \theta\Lambda_{n,T,i}(\theta|X_0^{n,T})d\theta, \quad i = 1, 2, 3.$$

The AMAPEi estimators are defined as

$$\hat{\theta}_{n,T,i} := \arg\max_{\theta\in\Theta} \Lambda_{n,T,i}(\theta|X_0^{n,T}), \quad i = 1, 2, 3.$$

If the parameter is assumed to be fixed but unknown, then there is no knowledge about the parameter, which is equivalent to assuming a non-informative improper prior. The above case then reduces to the familiar ML formulation.

In this paper, we obtain Berry-Esseen bounds for the estimators ABE$i := \tilde{\theta}_{n,T,i}$ and AMAPE$i := \hat{\theta}_{n,T,i}, \quad i = 1, 2, 3$.

Note that $\tilde{\theta}_{n,T,3}$ is not an approximate Bayes estimator in the true sense since $L_{n,T,3}$ is not an approximate likelihood, it is a pseudo likelihood. Nevertheless we keep the terminology. It is an approximate generalized Bayes estimator in the terminology in [18].

We obtain various rates of convergence to normality of the ABEs and AMAPEs using several normings. We also obtain stochastic bound on the difference between the ABEs and AMAPEs from their corresponding continuous counterparts.

We introduce a little bit of notations: Let $\Phi(\cdot)$ denote the standard normal distribution function. Throughout the paper C denotes a generic constant (it may depend on θ, but not on anything else). *Throughout the paper $\tilde{\theta}$ refers to posterior mean and $\hat{\theta}$ refers to posterior mode.*

Denote the fundamental martingale and the energy of the O-U process:

$$Z_T := \int_0^T X_t dW_t, \quad I_T := \int_0^T X_t^2 dt.$$

Let

$$Y_T := \int_0^T X_t dX_t, \quad I_{n,T} = \sum_{j=1}^n X_{t_{i-1}}^2 (t_j - t_{j-1}).$$

We need the following lemmas in the sequel.

Lemma 1.1 Let X, Y and Z be any three random variables on a probability space $(\Omega, \mathcal{F}, P)$ with $P(Z > 0) = 1$. Then, for any $\epsilon > 0$, we have

(a) $\sup_{x\in\mathbb{R}} |P\{X + Y \le x\} - \Phi(x)| \le \sup_{x\in\mathbb{R}} |P\{X \le x\} - \Phi(x)| + P(|Y| > \epsilon) + \epsilon.$

(b) $\sup_{x\in\mathbb{R}} |P\{\frac{X}{Z} \le x\} - \Phi(x)| \le \sup_{x\in\mathbb{R}} |P\{X \le x\} - \Phi(x)| + P\{|Z - 1| > \epsilon\} + \epsilon.$

Lemma 1.1 (a) is from [19] and proof of (b) is elementary. Proof of the following lemma is elementary.

Lemma 1.2 Let Q_n, R_n, Q and R be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ with $P(R_n > 0) = 1, P(R > 0) = 1$. Suppose $|Q_n - Q| = O_P(\delta_{1n})$ and $|R_n - R| = O_P(\delta_{2n})$ where $\delta_{1n}, \delta_{2n} \to 0$ as $n \to \infty$. Then,

$$\left|\frac{Q_n}{R_n} - \frac{Q}{R}\right| = O_P(\max(\delta_{1n}, \delta_{2n})).$$

Lemma 1.3 ([20])

$$\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{-2\theta}{T}\right)^{1/2} Z_T \le x \right\} - \Phi(x) \right| = O(T^{-1/2}).$$

Lemma 1.4 (a) ([21]) For $\epsilon = \epsilon(T) = o(1)$,

$$P\left\{ \left| \frac{-2\theta}{T} I_T - 1 \right| \ge \epsilon \right\} \le C e^{-T\epsilon^2}.$$

(b) ([6]) For every $\delta > 0$,

$$P\left\{ \left| \frac{-2\theta}{T} I_T - 1 \right| \ge \delta \right\} \le C T^{-1} \delta^{-2}.$$

Lemma 1.5 ([6])

$$\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{-2\theta}{T}\right)^{1/2} \left(\theta I_T - \frac{T}{2}\right) \le x \right\} - \Phi(x) \right| \le C T^{-1/2}.$$

The following lemma gives a bound on the difference of the discrete energy and the continuous energy of the O-U process.

Lemma 1.6 ([15])

$$E|I_{n,T} - I_T|^2 = O\left(\frac{T^4}{n^2}\right).$$

2. Berry-Esseen Bounds for the ABE1 and the AMAPE1

The following theorem gives the bound on the error of normal approximation of the ABEs and AMAPEs. Note that parts (a) and (d) use parameter dependent nonrandom norming. While this is useful for testing hypotheses about θ, it may not necessarily give a confidence interval. The normings in parts (b) (c), (e), and (f) are sample dependent which can be used for obtaining a confidence interval.

Theorem 2.1 Denote $\alpha_{n,T} := O(\max(T^{-1/2}(\log T)^{1/2},\ (\frac{T^3}{n^2})(\log T)^{-1}))$.

$$\text{(a)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| = O(\alpha_{n,T}),$$

$$\text{(b)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\tilde{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| = O(\alpha_{n,T}),$$

$$\text{(c)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\tilde{\theta}_{n,T,1}|}\right)^{1/2}(\tilde{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| = O(\alpha_{n,T}),$$

$$\text{(d)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| = O(\alpha_{n,T}),$$

$$\text{(e)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| = O(\alpha_{n,T}),$$

$$\text{(f)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\hat{\theta}_{n,T,1}|}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| = O(\alpha_{n,T}).$$

Proof (a) Observe that

$$\left(-\frac{T}{2\theta}\right)^{1/2}(\theta_{T,1}-\theta) = \frac{\left(-\frac{2\theta}{T}\right)^{1/2} Z_T}{\left(-\frac{2\theta}{T}\right) I_T} \tag{2.1}$$

Thus, we have using Theorem 5.3 (a),

$$I_{n,T}\tilde{\theta}_{n,T,1} = -\frac{T}{2} = Y_T + \theta I_T.$$

Hence, using Theorem 5.3 (a),

$$\begin{aligned}
&\left(-\frac{T}{2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,1}-\theta)\\
=&\ \frac{\left(-\frac{T}{2\theta}\right)^{1/2}Y_T+\theta\left(-\frac{T}{2\theta}\right)^{1/2}(I_T-I_{n,T})}{I_{n,T}}\\
=&\ \frac{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})}{\left(-\frac{2\theta}{T}\right)I_{n,T}} \qquad (2.2)
\end{aligned}$$

Further,

$$\begin{aligned}
&P\left\{\left|\left(\frac{-2\theta}{T}\right)(I_{n,T}-1)\right|>\epsilon\right\}\\
=&\ \left\{\left|\left(\frac{-2\theta}{T}\right)(I_{n,T}-I_T+I_T)-1\right|>\epsilon\right\}\\
\le&\ P\left\{\left|\left(\frac{-2\theta}{T}\right)I_T-1\right|>\frac{\epsilon}{2}\right\}+P\left\{\left(-\frac{2\theta}{T}\right)|I_{n,T}-I_T|>\frac{\epsilon}{2}\right\}\\
\le&\ C\exp\left(\frac{T\theta}{16}\epsilon^2\right)+\frac{16\theta^2}{T^2}\frac{E|I_{n,T}-I_T|^2}{\epsilon^2}\\
\le&\ C\exp\left(\frac{T\theta}{16}\epsilon^2\right)+C\frac{T^2/n^2}{\epsilon^2}. \qquad (2.3)
\end{aligned}$$

Next, observe that using Theorem 5.3 (a)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right|\\
=&\ \sup_{x\in\mathbb{R}}\left|P\left\{\frac{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})}{\left(-\frac{2\theta}{T}\right)I_{n,T}}\le x\right\}-\Phi(x)\right|\\
\le&\ \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right|+P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|>\epsilon\right\}\\
&+P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\epsilon\right\}+2\epsilon\\
\le&\ CT^{-1/2}+\theta^2\frac{\left(-\frac{2\theta}{T}\right)E|I_{n,T}-I_T|^2}{\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T^2}{n^2\epsilon^2}+2\epsilon \qquad (2.4)
\end{aligned}$$

(the bound for the 3rd term in the right hand side of (2.4) is obtained from (2.3))

$$\le CT^{-1/2}+C\frac{T^2}{n^2\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T}{n^2\epsilon^2}+\epsilon \qquad (2.5)$$

(by Lemma 1.5).

Choosing $\epsilon=CT^{-1/2}(\log T)^{1/2}$, the terms in the right hand side of (2.5) are of the order $O(\max(T^{-1/2}(\log T)^{1/2},\ (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(b) From (2.1), we have using Theorem 5.3 (a)

$$I_{n,T}^{1/2}(\tilde{\theta}_{n,T,1}-\theta) = \frac{Y_T + \theta(I_T - I_{n,T})}{I_{n,T}^{1/2}}.$$

Then, using Theorem 5.3 (a),

$$\begin{aligned}
& \sup_{x\in\mathbb{R}} \left| P\left\{ I_{n,T}^{1/2}(\tilde{\theta}_{n,T,1}-\theta) \le x \right\} - \Phi(x) \right| \\
= & \sup_{x\in\mathbb{R}} \left| P\left\{ \frac{Y_T}{I_{n,T}^{1/2}} + \theta \frac{I_T - I_{n,T}}{I_{n,T}^{1/2}} \le x \right\} - \Phi(x) \right| \\
\le & \sup_{x\in\mathbb{R}} \left| P\left\{ \frac{Y_T}{I_{n,T}^{1/2}} \le x \right\} - \Phi(x) \right| + P\left\{ \left| \frac{\theta(I_T - I_{n,T})}{I_{n,T}^{1/2}} \right| > \epsilon \right\} + \epsilon \\
=: & \ U_1 + U_2 + \epsilon. \qquad (2.6)
\end{aligned}$$

We have from (2.3),

$$U_1 \le CT^{-1/2} + C\exp\left(\frac{T\theta}{16}\epsilon^2\right) + C\frac{T^2}{n^2\epsilon^2} + \epsilon. \qquad (2.7)$$

Further,

$$\begin{aligned}
U_2 &= P\left\{ |\theta| \left| \frac{I_{n,T} - I_T}{I_{n,T}^{1/2}} \right| > \epsilon \right\} \\
&= P\left\{ |\theta| \frac{\left| \left(-\frac{2\theta}{T}\right)^{1/2} (I_{n,T} - I_T) \right|}{\left| \left(-\frac{2\theta}{T}\right)^{1/2} I_{n,T}^{1/2} \right|} > \epsilon \right\} \\
&\le P\left\{ \left| \left(-\frac{2\theta}{T}\right)^{1/2} \right| |I_{n,T} - I_T| > \delta \right\} + P\left\{ \left| \left(-\frac{2\theta}{T}\right)^{1/2} I_{n,T}^{1/2} - 1 \right| > \delta_1 \right\} \qquad (2.8) \\
& \quad \text{(where } \delta = \epsilon - C\epsilon^2 \text{ and } \delta_1 = (\epsilon - \delta)/\epsilon > 0) \\
&\le \left(-\frac{2\theta}{T}\right) \frac{E|I_{n,T} - I_T|^2}{\delta^2} + P\left\{ \left| \left(-\frac{2\theta}{T}\right) I_{n,T} - 1 \right| > \delta_1 \right\} \\
&\le C\frac{T^3}{n^2\delta^2} + C\exp\left(\frac{T\theta}{16}\delta_1^2\right) + C\frac{T^2}{n^2\delta_1^2}. \qquad (2.9)
\end{aligned}$$

Here, the bound for the first term in the right hand side of (2.7) comes from Lemma 1.5 and that for the second term is obtained from (2.3).

Now, using the bounds (2.7) and (2.9) in (2.6) with $\epsilon = CT^{-1/2}(\log T)^{1/2}$, we obtain that the terms in (2.6) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(c) Let

$$F_T := \left\{ |\tilde{\theta}_{n,T,1} - \theta| \le CT^{-1/2}(\log T)^{1/2} \right\}.$$

On the set F_T, expanding $(2|\tilde{\theta}_{n,T,1}|)^{1/2}$, using Theorem 5.3 (a), we obtain

$$(-2\tilde{\theta}_{n,T,1})^{-1/2} = (-2\theta)^{1/2}\left[1 - \frac{\theta - \tilde{\theta}_{n,T,1}}{\theta}\right]^{-1/2}$$

$$= (-2\theta)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\theta - \tilde{\theta}_{n,T,1}}{\theta}\right)\right] + O(T^{-1}(\log T)). \tag{2.10}$$

Then, using Theorem 5.3 (a)

$$\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\tilde{\theta}_{n,T,1}|}\right)^{1/2}(\tilde{\theta}_{n,T,1} - \theta) \le x\right\} - \Phi(x)\right|$$
$$\le \sup_{x\in\mathbb{R}}\left\{P\left(\frac{T}{2|\tilde{\theta}_{n,T,1}|}\right)^{1/2}(\tilde{\theta}_{n,T,1} - \theta) \le x,\ F_T\right\} + P(F_T^c).$$

Further, using Theorem 5.3 (a)

$$\begin{aligned}
P(F_T^c) &= P\left\{|\tilde{\theta}_{n,T,1} - \theta| > CT^{-1/2}(\log T)^{1/2}\right\} \\
&= P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}|\tilde{\theta}_{n,T,1} - \theta| > C(\log T)^{1/2}(-2\theta)^{-1/2}\right\} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^3}{n^2}(\log T)^{-1}\right) + 2(1 - \Phi(\log T^{1/2}(-2\theta)^{-1/2})) \\
&\quad \text{(by Theorem 2.1(a))} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^3}{n^2}(\log T)^{-1}\right).
\end{aligned}$$

On the set F_T, using Theorem 5.3 (a), we have

$$\left|\left(\frac{\tilde{\theta}_{n,T,1}}{\theta}\right)^{1/2} - 1\right| \le CT^{-1/2}(\log T)^{1/2}.$$

Hence, upon choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}, C$ large, using Theorem 5.3 (a) we obtain

$$\begin{aligned}
&\left| P\left\{ \left(\frac{T}{-2\tilde{\theta}_{n,T,1}} \right)^{1/2} (\tilde{\theta}_{n,T,1} - \theta) \leq x, \quad F_T \right\} - \Phi(x) \right| \\
\leq\ & \left| P\left\{ \left(\frac{T}{-2\theta} \right)^{1/2} (\tilde{\theta}_{n,T,1} - \theta) \leq x, \quad F_T \right\} \right| + P\left\{ \left| \left(\frac{\tilde{\theta}_{n,T,1}}{\theta} \right)^{1/2} - 1 \right| > \epsilon, \quad F_T \right\} + \epsilon \\
& \text{(by Lemma 1.1(b))} \\
\leq\ & C \max\left(T^{-1/2}(\log T)^{1/2}, \ \frac{T^4}{n^2}(\log T)^{-1} \right) \\
& \text{(by Theorem 2.1(a)).} \quad \square
\end{aligned}$$

(d) Observe that

$$\left(-\frac{T}{2\theta} \right)^{1/2} (\theta_{T,1} - \theta) = \frac{\left(\frac{-2\theta}{T}\right)^{1/2} Z_T}{\left(\frac{-2\theta}{T}\right) I_T} \tag{2.11}$$

Thus, we have

$$I_{n,T}\hat{\theta}_{n,T,1} = -\frac{T}{2} = Y_T + \theta I_T.$$

Hence, using Theorem 5.4 (a)

$$\begin{aligned}
&\left(-\frac{T}{2\theta} \right)^{1/2} (\hat{\theta}_{n,T,1} - \theta) \\
=\ & \frac{\left(-\frac{T}{2\theta}\right)^{1/2} Y_T + \theta \left(-\frac{T}{2\theta}\right)^{1/2} (I_T - I_{n,T})}{I_{n,T}} \\
=\ & \frac{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \left(-\frac{2\theta}{T}\right)^{1/2} (I_T - I_{n,T})}{\left(-\frac{2\theta}{T}\right) I_{n,T}} \qquad (2.12)
\end{aligned}$$

Further,

$$\begin{aligned}
& P\left\{ \left| \left(\frac{-2\theta}{T} \right) (I_{n,T} - 1) \right| > \epsilon \right\} \\
=\ & \left\{ \left| \left(\frac{-2\theta}{T} \right) (I_{n,T} - I_T + I_T) - 1 \right| > \epsilon \right\} \\
\leq\ & P\left\{ \left| \left(\frac{-2\theta}{T} \right) I_T - 1 \right| > \frac{\epsilon}{2} \right\} + P\left\{ \left(-\frac{2\theta}{T} \right) |I_{n,T} - I_T| > \frac{\epsilon}{2} \right\} \\
\leq\ & C \exp\left(\frac{T\theta}{16}\epsilon^2 \right) + \frac{16\theta^2}{T^2} \frac{E|I_{n,T} - I_T|^2}{\epsilon^2} \\
\leq\ & C \exp\left(\frac{T\theta}{16}\epsilon^2 \right) + C\frac{T^2/n^2}{\epsilon^2}. \qquad (2.13)
\end{aligned}$$

Next, using Theorem 5.4 (a), observe that

$$\begin{aligned}
& \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| \\
= & \sup_{x\in\mathbb{R}}\left|P\left\{\frac{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})}{\left(-\frac{2\theta}{T}\right)I_{n,T}}\le x\right\}-\Phi(x)\right| \\
\le & \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right|+P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|>\epsilon\right\} \\
& +P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\epsilon\right\}+2\epsilon \\
\le & CT^{-1/2}+\theta^2\frac{\left(-\frac{2\theta}{T}\right)E|I_{n,T}-I_T|^2}{\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T^2}{n^2\epsilon^2}+2\epsilon \qquad (2.14)
\end{aligned}$$

(the bound for the 3rd term in the right hand side of (2.14) is obtained from (2.13))

$$\le CT^{-1/2}+C\frac{T^2}{n^2\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T}{n^2\epsilon^2}+\epsilon \qquad (2.15)$$

(by Lemma 1.5).

Choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}$, the terms in the right hand side of (2.15) are of the order
$O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(e) From (2.11), using Theorem 5.4 (a), we have

$$I_{n,T}^{1/2}(\hat{\theta}_{n,T,1}-\theta)=\frac{Y_T+\theta(I_T-I_{n,T})}{I_{n,T}^{1/2}}.$$

Then,

$$\begin{aligned}
& \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| \\
= & \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}+\theta\frac{I_T-I_{n,T}}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right| \\
\le & \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right|+P\left\{\left|\frac{\theta(I_T-I_{n,T})}{I_{n,T}^{1/2}}\right|>\epsilon\right\}+\epsilon \\
=: & V_1+V_2+\epsilon. \qquad (2.16)
\end{aligned}$$

We have from (2.13),

$$V_1\le CT^{-1/2}+C\exp\left(\frac{T\theta}{16}\epsilon^2\right)+C\frac{T^2}{n^2\epsilon^2}+\epsilon. \qquad (2.17)$$

Further,

$$
\begin{aligned}
V_2 &= P\left\{|\theta|\left|\frac{I_{n,T}-I_T}{I_{n,T}^{1/2}}\right|>\epsilon\right\} \\
&= P\left\{|\theta|\frac{\left|\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|}{\left|\left(-\frac{2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}\right|}>\epsilon\right\} \\
&\le P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}\right||I_{n,T}-I_T|>\delta\right\}+P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}-1\right|>\delta_1\right\} \quad (2.18)\\
&\quad \text{(where } \delta=\epsilon-C\epsilon^2 \text{ and } \delta_1=(\epsilon-\delta)/\epsilon>0) \\
&\le \left(-\frac{2\theta}{T}\right)\frac{E|I_{n,T}-I_T|^2}{\delta^2}+P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\delta_1\right\} \\
&\le C\frac{T^3}{n^2\delta^2}+C\exp\left(\frac{T\theta}{16}\delta_1^2\right)+C\frac{T^2}{n^2\delta_1^2}. \quad (2.19)
\end{aligned}
$$

Here, the bound for the first term in the right hand side of (2.17) comes from Lemma 1.5 and that for the second term is obtained from (2.13).

Now, using the bounds (2.17) and (2.19) in (2.16) with $\epsilon = CT^{-1/2}(\log T)^{1/2}$, we obtain that the terms in (2.16) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(f) Let

$$F_{T,1} := \left\{|\hat{\theta}_{n,T,1}-\theta| \le CT^{-1/2}(\log T)^{1/2}\right\}.$$

On the set $F_{T,1}$, expanding $(2|\hat{\theta}_{n,T,1}|)^{1/2}$, using Theorem 5.4 (a), we obtain

$$(-2\hat{\theta}_{n,T,1})^{-1/2} = (-2\theta)^{1/2}\left[1-\frac{\theta-\hat{\theta}_{n,T,1}}{\theta}\right]^{-1/2}$$

$$= (-2\theta)^{1/2}\left[1+\frac{1}{2}\left(\frac{\theta-\hat{\theta}_{n,T,1}}{\theta}\right)\right]+O(T^{-1}(\log T)).$$

Then, using Theorem 5.4 (a)

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\hat{\theta}_{n,T,1}|}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| \\
&\le \sup_{x\in\mathbb{R}}\left\{P\left(\frac{T}{2|\hat{\theta}_{n,T}|}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x, F_{T,1}\right\}+P(F_{T,1}^c).
\end{aligned}
$$

Further, using Theorem 5.4 (a)

$$\begin{aligned} P(F_{T,1}^c) &= P\left\{|\hat{\theta}_{n,T,1}-\theta| > CT^{-1/2}(\log T)^{1/2}\right\} \\ &= P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}|\hat{\theta}_{n,T,1}-\theta| > C(\log T)^{1/2}(-2\theta)^{-1/2}\right\} \\ &\leq C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^3}{n^2}(\log T)^{-1}\right) + 2(1-\Phi(\log T^{1/2}(-2\theta)^{-1/2})) \\ &\quad \text{(by Theorem 2.1(a))} \\ &\leq C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^3}{n^2}(\log T)^{-1}\right). \end{aligned}$$

On the set $F_{T,1}$, we have

$$\left|\left(\frac{\hat{\theta}_{n,T,1}}{\theta}\right)^{1/2} - 1\right| \leq CT^{-1/2}(\log T)^{1/2}.$$

Hence, using Theorem 5.4 (a), upon choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}, C$ large, we obtain

$$\begin{aligned} &\left|P\left\{\left(\frac{T}{-2\hat{\theta}_{n,T,1}}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta) \leq x, \quad F_{T,1}\right\} - \Phi(x)\right| \\ \leq\ &\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta) \leq x, \quad F_{T,1}\right\}\right| + P\left\{\left|\left(\frac{\hat{\theta}_{n,T,1}}{\theta}\right)^{1/2} - 1\right| > \epsilon, \quad F_{T,1}\right\} + \epsilon \\ &\text{(by Lemma 1.1(b))} \\ \leq\ &C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^4}{n^2}(\log T)^{-1}\right) \\ &\text{(by Theorem 2.1(a)).} \quad \square \end{aligned}$$

The following theorem improves the Berry-Esseen bound in Theorem 2.1 using mixed norming.

Theorem 2.2

$$\text{(a)} \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\tilde{\theta}_{n,T,1}-\theta) \leq x\right\} - \Phi(x)\right| = O\left(\max\left(T^{-1/2}, \left(\frac{T^3}{n^2}\right)^{1/3}\right)\right),$$

$$\text{(b)} \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta) \leq x\right\} - \Phi(x)\right| = O\left(\max\left(T^{-1/2}, \left(\frac{T^3}{n^2}\right)^{1/3}\right)\right).$$

Proof (a) From (2.2), using Theorem 5.3 (a), we have

$$I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\tilde{\theta}_{n,T,1}-\theta) = \left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T}).$$

Hence, by Lemma 2.1(b), Lemma 2.2(c), and Theorem 5.3 (a)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\tilde{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| \\
=\ &\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\le x\right\}-\Phi(x)\right| \\
\le\ &\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right|+P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\right|>\epsilon\right\}+\epsilon \\
\le\ &CT^{-1/2}+C\frac{E|I_T-I_{n,T}|^2}{T\epsilon^2}+\epsilon\le CT^{-1/2}+C\frac{T^3}{n^2\epsilon^2}+\epsilon.
\end{aligned}$$

Choosing $\epsilon=(\frac{T^3}{n^2})^{1/3}$, the theorem follows. □

(b) From (2.2), using Theorem 5.4 (a), we have

$$I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta)=\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T}).$$

Hence, by Lemma 2.1(b), Lemma 2.2(c) and using Theorem 5.4 (a),

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,1}-\theta)\le x\right\}-\Phi(x)\right| \\
=\ &\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\le x\right\}-\Phi(x)\right| \\
\le\ &\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right|+P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\right|>\epsilon\right\}+\epsilon \\
\le\ &CT^{-1/2}+C\frac{E|I_T-I_{n,T}|^2}{T\epsilon^2}+\epsilon\le CT^{-1/2}+C\frac{T^3}{n^2\epsilon^2}+\epsilon.
\end{aligned}$$

Choosing $\epsilon=(\frac{T^3}{n^2})^{1/3}$, the theorem follows. □

3. Berry-Esseen Bounds for the ABE2 and the AMAPE2

The following theorem shows that ABE2 and AMAPE2 have sharper Berry-Esseen bounds than ABE1 and AMAPE1.

Theorem 3.1 Denote $\beta_{n,T}:=O(\max(T^{-1/2}(\log T)^{1/2},\ (\frac{T^4}{n^2})(\log T)^{-1}))$.

(a) $\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|=O(\beta_{n,T}),$

(b) $\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|=O(\beta_{n,T}),$

(c) $\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\tilde{\theta}_{n,T,2}|}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|=O(\beta_{n,T}).$

(d) $\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|=O(\beta_{n,T}),$

(e) $\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|=O(\beta_{n,T}),$

(f) $\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\hat{\theta}_{n,T,2}|}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|=O(\beta_{n,T}).$

Proof (a) Observe that

$$\left(-\frac{T}{2\theta}\right)^{1/2}(\theta_{T,2}-\theta)=\frac{\left(\frac{-2\theta}{T}\right)^{1/2}Z_T}{\left(\frac{-2\theta}{T}\right)I_T} \tag{3.1}$$

Thus, using Theorem 5.3 (b), we have

$$I_{n,T}\tilde{\theta}_{n,T,2}=-\frac{T}{2}=Y_T+\theta I_T.$$

Hence, using Theorem 5.3 (b),

$$\begin{aligned}&\left(-\frac{T}{2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\\ =\;&\frac{\left(-\frac{T}{2\theta}\right)^{1/2}Y_T+\theta\left(-\frac{T}{2\theta}\right)^{1/2}(I_T-I_{n,T})}{I_{n,T}}\\ =\;&\frac{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})}{\left(-\frac{2\theta}{T}\right)I_{n,T}}\end{aligned} \tag{3.2}$$

Next, using Theorem 5.3 (b), observe that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|\\
=&\sup_{x\in\mathbb{R}}\left|P\left\{\frac{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})}{\left(-\frac{2\theta}{T}\right)I_{n,T}}\le x\right\}-\Phi(x)\right|\\
\le&\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right|+P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|>\epsilon\right\}\\
&+P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\epsilon\right\}+2\epsilon\\
\le&\ CT^{-1/2}+\theta^2\frac{\left(-\frac{2\theta}{T}\right)E|I_{n,T}-I_T|^2}{\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T^2}{n^2\epsilon^2}+2\epsilon \quad (3.3)\\
&\text{(the bound for the 3}^{\text{rd}}\text{ term in the r.h.s. of (3.3) is obtained from (2.3))}\\
\le&\ CT^{-1/2}+C\frac{T^2}{n^2\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T}{n^2\epsilon^2}+\epsilon \quad (3.4)\\
&\text{(by Lemma 1.5).}
\end{aligned}$$

Choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}$, the terms in the right hand side of (3.5) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(b) From (3.1), using Theorem 5.3 (b), we have

$$I_{n,T}^{1/2}(\tilde{\theta}_{n,T,2}-\theta)=\frac{Y_T+\theta(I_T-I_{n,T})}{I_{n,T}^{1/2}}.$$

Then, using Theorem 5.3 (b),

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|\\
=&\sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}+\theta\frac{I_T-I_{n,T}}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right|\\
\le&\sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right|+P\left\{\left|\frac{\theta(I_T-I_{n,T})}{I_{n,T}^{1/2}}\right|>\epsilon\right\}+\epsilon\\
=:&\ J_1+J_2+\epsilon. \quad (3.5)
\end{aligned}$$

We have from (3.3),

$$J_1\le CT^{-1/2}+C\exp\left(\frac{T\theta}{16}\epsilon^2\right)+C\frac{T^2}{n^2\epsilon^2}+\epsilon. \quad (3.6)$$

Further,

$$
\begin{aligned}
J_2 &= P\left\{|\theta|\left|\frac{I_{n,T}-I_T}{I_{n,T}^{1/2}}\right|>\epsilon\right\} = P\left\{|\theta|\frac{\left|\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|}{\left|\left(-\frac{2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}\right|}>\epsilon\right\}\\
&\le P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}\right||I_{n,T}-I_T|>\delta\right\}+P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}-1\right|>\delta_1\right\} \qquad (3.7)\\
&\quad (\text{where } \delta=\epsilon-C\epsilon^2 \text{ and } \delta_1=(\epsilon-\delta)/\epsilon>0)\\
&\le \left(-\frac{2\theta}{T}\right)\frac{E|I_{n,T}-I_T|^2}{\delta^2}+P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\delta_1\right\}\\
&\le C\frac{T^3}{n^2\delta^2}+C\exp\left(\frac{T\theta}{16}\delta_1^2\right)+C\frac{T^2}{n^2\delta_1^2}. \qquad (3.8)
\end{aligned}
$$

Here, the bound for the first term in the right hand side of (3.6) comes from Lemma 1.5 and that for the second term is obtained from (2.3).

Now, using the bounds (3.6) and (3.8) in (3.5) with $\epsilon = CT^{-1/2}(\log T)^{1/2}$, we obtain that the terms in (3.5) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(c) Let

$$G_T := \left\{|\tilde\theta_{n,T,2}-\theta| \le CT^{-1/2}(\log T)^{1/2}\right\}.$$

On the set G_T, expanding $(2|\tilde\theta_{n,T,2}|)^{1/2}$, using Theorem 5.3 (b), we obtain

$$(-2\tilde\theta_{n,T,2})^{-1/2} = (-2\theta)^{1/2}\left[1-\frac{\theta-\tilde\theta_{n,T,2}}{\theta}\right]^{-1/2}$$

$$= (-2\theta)^{1/2}\left[1+\frac{1}{2}\left(\frac{\theta-\tilde\theta_{n,T,2}}{\theta}\right)\right]+O(T^{-1}(\log T)).$$

Then, using Theorem 5.3 (b),

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\tilde\theta_{n,T,2}|}\right)^{1/2}(\tilde\theta_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right|\\
\le\ &\sup_{x\in\mathbb{R}}\left\{P\left(\frac{T}{2|\tilde\theta_{n,T,2}|}\right)^{1/2}(\tilde\theta_{n,T,2}-\theta)\le x,\quad G_T\right\}+P(G_T^c).
\end{aligned}
$$

Further, using Theorem 5.3 (b),

$$\begin{aligned} P(G_T^c) &= P\left\{|\tilde{\theta}_{n,T,2}-\theta| > CT^{-1/2}(\log T)^{1/2}\right\} \\ &= P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}|\tilde{\theta}_{n,T,2}-\theta| > C(\log T)^{1/2}(-2\theta)^{-1/2}\right\} \\ &\leq C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^3}{n^2}(\log T)^{-1}\right) + 2(1-\Phi(\log T^{1/2}(-2\theta)^{-1/2})) \\ &\quad \text{(by Theorem 3.1(a))} \\ &\leq C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^3}{n^2}(\log T)^{-1}\right). \end{aligned}$$

On the set G_T, using Theorem 5.3 (b),

$$\left|\left(\frac{\tilde{\theta}_{n,T,2}}{\theta}\right)^{1/2} - 1\right| \leq CT^{-1/2}(\log T)^{1/2}.$$

Hence, upon choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}, C$ large, using Theorem 5.3 (b), we obtain

$$\begin{aligned} &\left|P\left\{\left(\frac{T}{-2\tilde{\theta}_{n,T,2}}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta) \leq x,\ G_T\right\} - \Phi(x)\right| \\ \leq\ &\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta) \leq x, G_T\right\}\right| + P\left\{\left|\left(\frac{\tilde{\theta}_{n,T,2}}{\theta}\right)^{1/2} - 1\right| > \epsilon,\ G_T\right\} + \epsilon \\ &\text{(by Lemma 1.1(b))} \\ \leq\ &C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^4}{n^2}(\log T)^{-1}\right) \\ &\text{(by Theorem 3.1(a)).} \quad \square \end{aligned}$$

(d) Observe that

$$\left(-\frac{T}{2\theta}\right)^{1/2}(\theta_{T,2}-\theta) = \frac{\left(\frac{-2\theta}{T}\right)^{1/2} Z_T}{\left(\frac{-2\theta}{T}\right) I_T} \tag{3.9}$$

Thus, using Theorem 5.3 (b), we have

$$I_{n,T}\hat{\theta}_{n,T,2} = -\frac{T}{2} = Y_T + \theta I_T.$$

Hence, using Theorem 5.3 (b),

$$\begin{aligned}
&\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta) \\
= \quad & \frac{\left(-\frac{T}{2\theta}\right)^{1/2} Y_T + \theta\left(-\frac{T}{2\theta}\right)^{1/2}(I_T - I_{n,T})}{I_{n,T}} \\
= \quad & \frac{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T})}{\left(-\frac{2\theta}{T}\right) I_{n,T}} \qquad (3.10)
\end{aligned}$$

Next, using Theorem 5.3 (b), observe that

$$\begin{aligned}
& \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right| \\
= \quad & \sup_{x\in\mathbb{R}}\left|P\left\{\frac{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T})}{\left(-\frac{2\theta}{T}\right) I_{n,T}}\le x\right\}-\Phi(x)\right| \\
\le \quad & \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T\le x\right\}-\Phi(x)\right| + P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|>\epsilon\right\} \\
& +P\left\{\left|\left(-\frac{2\theta}{T}\right) I_{n,T}-1\right|>\epsilon\right\}+2\epsilon \\
\le \quad & CT^{-1/2}+\theta^2\frac{\left(-\frac{2\theta}{T}\right)E|I_{n,T}-I_T|^2}{\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T^2}{n^2\epsilon^2}+2\epsilon \quad (3.11) \\
& \text{(the bound for the 3}^{\text{rd}}\text{ term in the r.h.s. of (3.11) is obtained from (2.3))} \\
\le \quad & CT^{-1/2}+C\frac{T^2}{n^2\epsilon^2}+C\exp\left(\frac{T\theta}{4}\epsilon^2\right)+C\frac{T}{n^2\epsilon^2}+\epsilon \qquad (3.12) \\
& \text{(by Lemma 1.5).}
\end{aligned}$$

Choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}$, the terms in the right hand side of (3.13) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(e) From (3.1), using Theorem 5.4 (b), we have

$$I_{n,T}^{1/2}(\hat{\theta}_{n,T,2}-\theta)=\frac{Y_T+\theta(I_T-I_{n,T})}{I_{n,T}^{1/2}}.$$

Then, using Theorem 5.4 (b)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right| \\
= \quad & \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}+\theta\frac{I_T-I_{n,T}}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right| \\
\le \quad & \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right|+P\left\{\left|\frac{\theta(I_T-I_{n,T})}{I_{n,T}^{1/2}}\right|>\epsilon\right\}+\epsilon \\
=: \quad & K_1+K_2+\epsilon. \qquad (3.13)
\end{aligned}$$

We have from (3.11),

$$K_1 \le CT^{-1/2}+C\exp\left(\frac{T\theta}{16}\epsilon^2\right)+C\frac{T^2}{n^2\epsilon^2}+\epsilon. \qquad (3.14)$$

Further,

$$\begin{aligned}
K_2 \quad = \quad & P\left\{|\theta|\left|\frac{I_{n,T}-I_T}{I_{n,T}^{1/2}}\right|>\epsilon\right\}=P\left\{|\theta|\frac{\left|\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|}{\left|\left(-\frac{2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}\right|}>\epsilon\right\} \\
\le \quad & P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}\right||I_{n,T}-I_T|>\delta\right\}+P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}-1\right|>\delta_1\right\} \qquad (3.15) \\
& \text{(where } \delta=\epsilon-C\epsilon^2 \text{ and } \delta_1=(\epsilon-\delta)/\epsilon>0) \\
\le \quad & \left(-\frac{2\theta}{T}\right)\frac{E|I_{n,T}-I_T|^2}{\delta^2}+P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\delta_1\right\} \\
\le \quad & C\frac{T^3}{n^2\delta^2}+C\exp\left(\frac{T\theta}{16}\delta_1^2\right)+C\frac{T^2}{n^2\delta_1^2}. \qquad (3.16)
\end{aligned}$$

Here, the bound for the first term in the right hand side of (3.14) comes from Lemma 1.5 and that for the second term is obtained from (2.3).

Now, using the bounds (3.14) and (3.16) in (3.13) with $\epsilon = CT^{-1/2}(\log T)^{1/2}$, we obtain that the terms in (3.13) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(f) Let

$$G_{T,1} := \left\{|\hat{\theta}_{n,T,2}-\theta| \le CT^{-1/2}(\log T)^{1/2}\right\}.$$

On the set $G_{T,1}$, expanding $(2|\hat{\theta}_{n,T,2}|)^{1/2}$, using Theorem 5.4 (b), we obtain

$$(-2\hat{\theta}_{n,T,2})^{-1/2} = (-2\theta)^{1/2}\left[1-\frac{\theta-\hat{\theta}_{n,T,2}}{\theta}\right]^{-1/2}$$

$$= (-2\theta)^{1/2}\left[1+\frac{1}{2}\left(\frac{\theta-\hat{\theta}_{n,T,2}}{\theta}\right)\right]+O(T^{-1}(\log T)).$$

Then, using Theorem 5.4 (b)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\hat{\theta}_{n,T,2}|}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right| \\
\le\ &\sup_{x\in\mathbb{R}}\left\{P\left(\frac{T}{2|\hat{\theta}_{n,T,2}|}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x,\quad G_{T,1}\right\}+P(G^c_{T,1}).
\end{aligned}$$

Further, using Theorem 5.4 (b),

$$\begin{aligned}
P(G^c_{T,1}) &= P\left\{|\hat{\theta}_{n,T,2}-\theta|>CT^{-1/2}(\log T)^{1/2}\right\} \\
&= P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}|\hat{\theta}_{n,T,2}-\theta|>C(\log T)^{1/2}(-2\theta)^{-1/2}\right\} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2},\frac{T^3}{n^2}(\log T)^{-1}\right)+2(1-\Phi((\log T)^{1/2}(-2\theta)^{-1/2})) \\
&\quad \text{(by Theorem 3.1(a))} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^3}{n^2}(\log T)^{-1}\right).
\end{aligned}$$

On the set $G_{T,1}$, using Theorem 5.4 (b)

$$\left|\left(\frac{\hat{\theta}_{n,T,2}}{\theta}\right)^{1/2}-1\right|\le CT^{-1/2}(\log T)^{1/2}.$$

Hence, upon choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}, C$ large, using Theorem 5.4 (b), we obtain

$$\begin{aligned}
&\left|P\left\{\left(\frac{T}{-2\hat{\theta}_{n,T,2}}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x,\ G_{T,1}\right\}-\Phi(x)\right| \\
\le\ &\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x,\ G_{T,1}\right\}\right|+P\left\{\left|\left(\frac{\hat{\theta}_{n,T,2}}{\theta}\right)^{1/2}-1\right|>\epsilon,\ G_{T,1}\right\}+\epsilon \\
&\text{(by Lemma 1.1(b))} \\
\le\ &C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^4}{n^2}(\log T)^{-1}\right) \\
&\text{(by Theorem 3.1(a)).}\quad\square
\end{aligned}$$

The following theorem improves the Berry-Esseen bound in Theorem 3.1 using mixed norming.

Theorem 3.2

$$\text{(a)} \sup_{x\in\mathbb{R}} \left| P\left\{ I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\le x\right\} - \Phi(x)\right| = O\left(\max\left(T^{-1/2}, \left(\frac{T^3}{n^2}\right)^{1/3}\right)\right).$$

$$\text{(b)} \sup_{x\in\mathbb{R}} \left| P\left\{ I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\} - \Phi(x)\right| = O\left(\max\left(T^{-1/2}, \left(\frac{T^3}{n^2}\right)^{1/3}\right)\right).$$

Proof (a) From (3.2), using Theorem 5.3 (b), we have

$$I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta) = \left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T}).$$

Hence, by Lemma 2.1(b), Lemma 1.5 and using Theorem 5.3 (b)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| P\left\{ I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\tilde{\theta}_{n,T,2}-\theta)\le x\right\} - \Phi(x)\right| \\
=\ &\sup_{x\in\mathbb{R}} \left| P\left\{ \left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T}) \le x\right\} - \Phi(x)\right| \\
\le\ &\sup_{x\in\mathbb{R}} \left| P\left\{ \left(-\frac{2\theta}{T}\right)^{1/2} Y_T \le x\right\} - \Phi(x)\right| + P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T})\right| > \epsilon\right\} + \epsilon \\
\le\ & CT^{-1/2} + C\frac{E|I_T - I_{n,T}|^2}{T\epsilon^2} + \epsilon \\
\le\ & CT^{-1/2} + C\frac{T^3}{n^2\epsilon^2} + \epsilon.
\end{aligned}$$

Choosing $\epsilon = (\frac{T^3}{n^2})^{1/3}$, the theorem follows. □

(b) From (3.2), using Theorem 5.4 (b), we have

$$I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta) = \left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T}).$$

Hence, by Lemma 2.1(b), Lemma 1.5 and using Theorem 5.4 (b)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,2}-\theta)\le x\right\}-\Phi(x)\right| \\
= &\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\le x\right\}-\Phi(x)\right| \\
\le &\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right|+P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\right|>\epsilon\right\}+\epsilon \\
\le &\; CT^{-1/2}+C\frac{E|I_T-I_{n,T}|^2}{T\epsilon^2}+\epsilon \\
\le &\; CT^{-1/2}+C\frac{T^3}{n^2\epsilon^2}+\epsilon.
\end{aligned}$$

Choosing $\epsilon=(\frac{T^3}{n^2})^{1/3}$, the theorem follows. □

4. Berry-Esseen Bounds for the ABE3 and the AMAPE3

The following theorem shows that ABE3 has a sharper Berry-Esseen bound than ABE1.

Theorem 4.1 Denote $\gamma_{n,T}:=O(\max(T^{-1/2}(\log T)^{1/2},\ (\frac{T^4}{n^2})(\log T)^{-1}))$.

$$\begin{aligned}
&\text{(a)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right|=O(\gamma_{n,T}),\\
&\text{(b)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\tilde{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right|=O(\gamma_{n,T}),\\
&\text{(c)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\tilde{\theta}_{n,T,3}|}\right)^{1/2}(\tilde{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right|=O(\gamma_{n,T}).\\
&\text{(d)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right|=O(\gamma_{n,T}),\\
&\text{(e)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\hat{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right|=O(\gamma_{n,T}),\\
&\text{(f)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\hat{\theta}_{n,T,3}|}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right|=O(\gamma_{n,T}).
\end{aligned}$$

Proof (a) Observe that

$$\left(-\frac{T}{2\theta}\right)^{1/2}(\theta_{T,3}-\theta)=\frac{\left(\frac{-2\theta}{T}\right)^{1/2}Y_T}{\left(\frac{-2\theta}{T}\right)I_T} \tag{4.1}$$

where

$$Y_T := -\theta I_T - \frac{T}{2} \quad \text{and} \quad I_T := \int_0^T X_t^2 dt.$$

Thus, using Theorem 5.3 (c), we have,

$$I_{n,T}\tilde{\theta}_{n,T,3} = -\frac{T}{2} = Y_T + \theta I_T.$$

Hence, using Theorem 5.3 (c),

$$\begin{aligned}
&\left(-\frac{T}{2\theta}\right)^{1/2} (\tilde{\theta}_{n,T,3} - \theta) \\
= \;& \frac{\left(-\frac{T}{2\theta}\right)^{1/2} Y_T + \theta \left(-\frac{T}{2\theta}\right)^{1/2} (I_T - I_{n,T})}{I_{n,T}} \\
= \;& \frac{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \left(-\frac{2\theta}{T}\right)^{1/2} (I_T - I_{n,T})}{\left(-\frac{2\theta}{T}\right) I_{n,T}} \qquad (4.2)
\end{aligned}$$

Next, observe that using Theorem 5.3 (c),

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| P\left\{ \left(-\frac{T}{2\theta}\right)^{1/2} (\tilde{\theta}_{n,T,3} - \theta) \le x \right\} - \Phi(x) \right| \\
= \;& \sup_{x\in\mathbb{R}} \left| P\left\{ \frac{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \left(-\frac{2\theta}{T}\right)^{1/2} (I_T - I_{n,T})}{\left(-\frac{2\theta}{T}\right) I_{n,T}} \le x \right\} - \Phi(x) \right| \\
\le \;& \sup_{x\in\mathbb{R}} \left| P\left\{ \left(-\frac{2\theta}{T}\right)^{1/2} Y_T \le x \right\} - \Phi(x) \right| + P\left\{ \left| \theta \left(-\frac{2\theta}{T}\right)^{1/2} (I_{n,T} - I_T) \right| > \epsilon \right\} \\
&+ P\left\{ \left| \left(-\frac{2\theta}{T}\right) I_{n,T} - 1 \right| > \epsilon \right\} + 2\epsilon \\
\le \;& CT^{-1/2} + \theta^2 \frac{\left(-\frac{2\theta}{T}\right) E|I_{n,T} - I_T|^2}{\epsilon^2} + C\exp\left(\frac{T\theta}{4}\epsilon^2\right) + C\frac{T^2}{n^2\epsilon^2} + 2\epsilon \quad (4.3) \\
&\text{(the bound for the 3}^{\text{rd}}\text{ term in the r.h.s. of (4.3) is obtained from (2.3))} \\
\le \;& CT^{-1/2} + C\frac{T^2}{n^2\epsilon^2} + C\exp\left(\frac{T\theta}{4}\epsilon^2\right) + C\frac{T}{n^2\epsilon^2} + \epsilon \qquad (4.4) \\
&\text{(by Lemma 1.5).}
\end{aligned}$$

Choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}$, the terms in the right hand side of (4.4) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(b) From (4.1), using Theorem 5.3 (c), we have

$$I_{n,T}^{1/2}(\tilde{\theta}_{n,T,3} - \theta) = \frac{Y_T + \theta(I_T - I_{n,T})}{I_{n,T}^{1/2}}.$$

Then, using Theorem 5.3 (c),

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| P\left\{ I_{n,T}^{1/2}(\tilde{\theta}_{n,T,3}-\theta) \le x \right\} - \Phi(x) \right| \\
=& \sup_{x\in\mathbb{R}} \left| P\left\{ \frac{Y_T}{I_{n,T}^{1/2}} + \theta \frac{I_T - I_{n,T}}{I_{n,T}^{1/2}} \le x \right\} - \Phi(x) \right| \\
\le& \sup_{x\in\mathbb{R}} \left| P\left\{ \frac{Y_T}{I_{n,T}^{1/2}} \le x \right\} - \Phi(x) \right| + P\left\{ \left| \frac{\theta(I_T - I_{n,T})}{I_{n,T}^{1/2}} \right| > \epsilon \right\} + \epsilon \\
=:& \ M_1 + M_2 + \epsilon. \qquad (4.5)
\end{aligned}$$

We have from (2.3),

$$M_1 \le CT^{-1/2} + C\exp\left(\frac{T\theta}{16}\epsilon^2\right) + C\frac{T^2}{n^2\epsilon^2} + \epsilon. \tag{4.6}$$

Further,

$$\begin{aligned}
M_2 &= P\left\{ |\theta| \left| \frac{I_{n,T} - I_T}{I_{n,T}^{1/2}} \right| > \epsilon \right\} = P\left\{ |\theta| \frac{\left| \left(-\frac{2\theta}{T}\right)^{1/2} (I_{n,T} - I_T) \right|}{\left| \left(-\frac{2\theta}{T}\right)^{1/2} I_{n,T}^{1/2} \right|} > \epsilon \right\} \\
&\le P\left\{ \left| \left(-\frac{2\theta}{T}\right)^{1/2} \right| |I_{n,T} - I_T| > \delta \right\} + P\left\{ \left| \left(-\frac{2\theta}{T}\right)^{1/2} I_{n,T}^{1/2} - 1 \right| > \delta_1 \right\} \qquad (4.7) \\
&\quad \text{(where } \delta = \epsilon - C\epsilon^2 \text{ and } \delta_1 = (\epsilon-\delta)/\epsilon > 0) \\
&\le \left(-\frac{2\theta}{T}\right) \frac{E|I_{n,T} - I_T|^2}{\delta^2} + P\left\{ \left| \left(-\frac{2\theta}{T}\right) I_{n,T} - 1 \right| > \delta_1 \right\} \\
&\le C\frac{T^3}{n^2\delta^2} + C\exp\left(\frac{T\theta}{16}\delta_1^2\right) + C\frac{T^2}{n^2\delta_1^2}. \qquad (4.8)
\end{aligned}$$

Here, the bound for the first term in the right hand side of (4.6) comes from Lemma 1.5 and that for the second term is obtained from (2.3).

Now, using the bounds (4.6) and (4.5) in (4.6) with $\epsilon = CT^{-1/2}(\log T)^{1/2}$, we obtain that the terms in (4.5) are of the order $O(\max(T^{-1/2}(\log T)^{1/2},\ (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(c) Let

$$H_T := \left\{ |\tilde{\theta}_{n,T,3} - \theta| \le CT^{-1/2}(\log T)^{1/2} \right\}.$$

On the set H_T, expanding $(2|\tilde{\theta}_{n,T,3}|)^{1/2}$, using Theorem 5.3 (c), we obtain

$$(-2\tilde{\theta}_{n,T,3})^{-1/2} = (-2\theta)^{1/2}\left[1 - \frac{\theta - \tilde{\theta}_{n,T,3}}{\theta}\right]^{-1/2}$$

$$= (-2\theta)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\theta - \tilde{\theta}_{n,T,3}}{\theta}\right)\right] + O(T^{-1}(\log T)).$$

Then, using Theorem 5.3 (c),

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\tilde{\theta}_{n,T,3}|}\right)^{1/2}(\tilde{\theta}_{n,T,3}-\theta)\le x\right\} - \Phi(x)\right| \\
\le\ &\sup_{x\in\mathbb{R}}\left\{P\left(\frac{T}{2|\tilde{\theta}_{n,T,3}|}\right)^{1/2}(\tilde{\theta}_{n,T,3}-\theta)\le x,\ H_T\right\} + P(H_T^c).
\end{aligned}$$

Further, using Theorem 5.3 (c),

$$\begin{aligned}
P(H_T^c) &= P\left\{|\tilde{\theta}_{n,T,3}-\theta| > CT^{-1/2}(\log T)^{1/2}\right\} \\
&= P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}|\tilde{\theta}_{n,T,3}-\theta| > C(\log T)^{1/2}(-2\theta)^{-1/2}\right\} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^3}{n^2}(\log T)^{-1}\right) + 2(1-\Phi((\log T)^{1/2}(-2\theta)^{-1/2})) \\
&\quad \text{(by Theorem 4.1(a))} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^3}{n^2}(\log T)^{-1}\right).
\end{aligned}$$

On the set H_T, using Theorem 5.3 (c),

$$\left|\left(\frac{\tilde{\theta}_{n,T,3}}{\theta}\right)^{1/2} - 1\right| \le CT^{-1/2}(\log T)^{1/2}.$$

Hence, upon choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}, C$ large, using Theorem 5.3 (c), we obtain

$$\begin{aligned}
&\left|P\left\{\left(\frac{T}{-2\tilde{\theta}_{n,T,3}}\right)^{1/2}(\tilde{\theta}_{n,T,3}-\theta)\le x,\ H_T\right\} - \Phi(x)\right| \\
\le\ &\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\tilde{\theta}_{n,T,3}-\theta)\le x, H_T\right\}\right| + P\left\{\left|\left(\frac{\tilde{\theta}_{n,T,3}}{\theta}\right)^{1/2} - 1\right| > \epsilon,\ H_T\right\} + \epsilon \\
&\text{(by Lemma 1.1(b))} \\
\le\ &C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^4}{n^2}(\log T)^{-1}\right) \\
&\text{(by Theorem 4.1(a)).} \quad \square
\end{aligned}$$

(d) Observe that

$$\left(-\frac{T}{2\theta}\right)^{1/2}(\theta_{T,3}-\theta) = \frac{\left(\frac{-2\theta}{T}\right)^{1/2} Y_T}{\left(\frac{-2\theta}{T}\right) I_T} \tag{4.9}$$

where

$$Y_T := -\theta I_T - \frac{T}{2} \quad \text{and} \quad I_T := \int_0^T X_t^2 dt.$$

Thus, using Theorem 5.4 (c), we have

$$I_{n,T}\hat{\theta}_{n,T,3} = -\frac{T}{2} = Y_T + \theta I_T.$$

Hence, using Theorem 5.4 (c)

$$\begin{aligned}
&\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta) \\
= \; & \frac{\left(-\frac{T}{2\theta}\right)^{1/2} Y_T + \theta\left(-\frac{T}{2\theta}\right)^{1/2}(I_T - I_{n,T})}{I_{n,T}} \\
= \; & \frac{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T})}{\left(-\frac{2\theta}{T}\right) I_{n,T}} \qquad (4.10)
\end{aligned}$$

Next, using Theorem 5.4 (c), observe that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta)\le x\right\} - \Phi(x)\right| \\
= \; & \sup_{x\in\mathbb{R}}\left|P\left\{\frac{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T})}{\left(-\frac{2\theta}{T}\right) I_{n,T}} \le x\right\} - \Phi(x)\right| \\
\le \; & \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T \le x\right\} - \Phi(x)\right| + P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right| > \epsilon\right\} \\
& + P\left\{\left|\left(-\frac{2\theta}{T}\right) I_{n,T} - 1\right| > \epsilon\right\} + 2\epsilon \\
\le \; & CT^{-1/2} + \theta^2 \frac{\left(-\frac{2\theta}{T}\right) E|I_{n,T}-I_T|^2}{\epsilon^2} + C\exp\left(\frac{T\theta}{4}\epsilon^2\right) + C\frac{T^2}{n^2\epsilon^2} + 2\epsilon \qquad (4.11) \\
& \text{(the bound for the 3}^{\text{rd}}\text{ term in the r.h.s. of (4.11) is obtained from (2.3))} \\
\le \; & CT^{-1/2} + C\frac{T^2}{n^2\epsilon^2} + C\exp\left(\frac{T\theta}{4}\epsilon^2\right) + C\frac{T}{n^2\epsilon^2} + \epsilon \qquad (4.12) \\
& \text{(by Lemma 1.5).}
\end{aligned}$$

Choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}$, the terms in the right hand side of (2.5) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(e) From (4.1), using Theorem 5.4 (c), we have

$$I_{n,T}^{1/2}(\hat{\theta}_{n,T,3} - \theta) = \frac{Y_T + \theta(I_T - I_{n,T})}{I_{n,T}^{1/2}}.$$

Then, using Theorem 5.4 (c)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left| P\left\{ I_{n,T}^{1/2}(\hat{\theta}_{n,T,3} - \theta) \le x \right\} - \Phi(x) \right| \\
=\ & \sup_{x\in\mathbb{R}} \left| P\left\{ \frac{Y_T}{I_{n,T}^{1/2}} + \theta \frac{I_T - I_{n,T}}{I_{n,T}^{1/2}} \le x \right\} - \Phi(x) \right| \\
\le\ & \sup_{x\in\mathbb{R}} \left| P\left\{ \frac{Y_T}{I_{n,T}^{1/2}} \le x \right\} - \Phi(x) \right| + P\left\{ \left| \frac{\theta(I_T - I_{n,T})}{I_{n,T}^{1/2}} \right| > \epsilon \right\} + \epsilon \\
=:\ & N_1 + N_2 + \epsilon. \qquad (4.13)
\end{aligned}$$

We have from (2.3),

$$N_1 \le CT^{-1/2} + C\exp\left(\frac{T\theta}{16}\epsilon^2\right) + C\frac{T^2}{n^2\epsilon^2} + \epsilon. \tag{4.14}$$

Further,

$$\begin{aligned}
N_2 &= P\left\{ |\theta| \left| \frac{I_{n,T} - I_T}{I_{n,T}^{1/2}} \right| > \epsilon \right\} = P\left\{ |\theta| \frac{\left| \left(-\frac{2\theta}{T}\right)^{1/2} (I_{n,T} - I_T) \right|}{\left| \left(-\frac{2\theta}{T}\right)^{1/2} I_{n,T}^{1/2} \right|} > \epsilon \right\} \\
&\le P\left\{ \left| \left(-\frac{2\theta}{T}\right)^{1/2} \right| |I_{n,T} - I_T| > \delta \right\} + P\left\{ \left| \left(-\frac{2\theta}{T}\right)^{1/2} I_{n,T}^{1/2} - 1 \right| > \delta_1 \right\} \qquad (4.15) \\
&\quad \text{(where } \delta = \epsilon - C\epsilon^2 \text{ and } \delta_1 = (\epsilon - \delta)/\epsilon > 0) \\
&\le \left(-\frac{2\theta}{T}\right) \frac{E|I_{n,T} - I_T|^2}{\delta^2} + P\left\{ \left| \left(-\frac{2\theta}{T}\right) I_{n,T} - 1 \right| > \delta_1 \right\} \\
&\le C\frac{T^3}{n^2\delta^2} + C\exp\left(\frac{T\theta}{16}\delta_1^2\right) + C\frac{T^2}{n^2\delta_1^2}. \qquad (4.16)
\end{aligned}$$

Here, the bound for the first term in the right hand side of (4.14) comes from Lemma 1.5 and that for the second term is obtained from (2.3).

Now, using the bounds (4.14) and (4.13) in (4.14) with $\epsilon = CT^{-1/2}(\log T)^{1/2}$, we obtain that the terms in (4.13) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (\frac{T^4}{n^2})(\log T)^{-1}))$. □

(f) Let

$$H_{T,1} := \left\{ |\hat{\theta}_{n,T,3} - \theta| \le CT^{-1/2}(\log T)^{1/2} \right\}.$$

On the set $H_{T,1}$, using Theorem 5.4 (c), expanding $(2|\hat{\theta}_{n,T,3}|)^{1/2}$, we obtain

$$(-2\hat{\theta}_{n,T,3})^{-1/2} = (-2\theta)^{1/2}\left[1 - \frac{\theta - \hat{\theta}_{n,T,3}}{\theta}\right]^{-1/2}$$

$$= (-2\theta)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\theta - \hat{\theta}_{n,T,3}}{\theta}\right)\right] + O(T^{-1}(\log T)).$$

Then, using Theorem 5.4 (c),

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\hat{\theta}_{n,T,3}|}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right| \\
\le\ &\sup_{x\in\mathbb{R}}\left\{P\left(\frac{T}{2|\hat{\theta}_{n,T,3}|}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta)\le x,\ H_{T,1}\right\} + P(H^c_{T,1}).
\end{aligned}$$

Further, using Theorem 5.4 (c)

$$\begin{aligned}
P(H^c_{T,1}) &= P\left\{|\hat{\theta}_{n,T,3}-\theta| > CT^{-1/2}(\log T)^{1/2}\right\} \\
&= P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}|\hat{\theta}_{n,T,3}-\theta| > C(\log T)^{1/2}(-2\theta)^{-1/2}\right\} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2},\ \frac{T^3}{n^2}(\log T)^{-1}\right) + 2(1-\Phi((\log T)^{1/2}(-2\theta)^{-1/2})) \\
&\quad \text{(by Theorem 4.1(a))} \\
&\le C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^3}{n^2}(\log T)^{-1}\right).
\end{aligned}$$

On the set $H_{T,1}$, using Theorem 5.4 (c)

$$\left|\left(\frac{\hat{\theta}_{n,T,3}}{\theta}\right)^{1/2} - 1\right| \le CT^{-1/2}(\log T)^{1/2}.$$

Hence, upon choosing $\epsilon = CT^{-1/2}(\log T)^{1/2}, C$ large, using Theorem 5.4 (c), we obtain

$$\begin{aligned}
&\left| P\left\{ \left(\frac{T}{-2\hat{\theta}_{n,T,3}} \right)^{1/2} (\hat{\theta}_{n,T,3} - \theta) \le x,\ H_{T,1} \right\} - \Phi(x) \right| \\
\le\ & \left| P\left\{ \left(\frac{T}{-2\theta} \right)^{1/2} (\hat{\theta}_{n,T,3} - \theta) \le x,\ H_{T,1} \right\} \right| + P\left\{ \left| \left(\frac{\hat{\theta}_{n,T,3}}{\theta} \right)^{1/2} - 1 \right| > \epsilon,\ H_{T,1} \right\} + \epsilon \\
& \text{(by Lemma 1.1(b))} \\
\le\ & C \max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^4}{n^2}(\log T)^{-1} \right) \\
& \text{(by Theorem 4.1(a)).} \quad \square
\end{aligned}$$

In the following theorem, we improve the bound on the error of normal approximation using a mixture of random and nonrandom normings. Thus asymptotic normality of the ABE needs $T \to \infty$ and $\frac{T}{n^{2/3}} \to 0$ which are sharper than the bound in Theorem 4.1.

The following theorem improves the Berry-Esseen bounds in Theorem 4.1 using mixed norming.

Theorem 4.2

$$\text{(a)} \sup_{x \in \mathbb{R}} \left| P\left\{ I_{n,T} \left(-\frac{2\theta}{T} \right)^{1/2} (\tilde{\theta}_{n,T,3} - \theta) \le x \right\} - \Phi(x) \right| = O\left(\max\left(T^{-1/2}, \left(\frac{T^3}{n^2} \right)^{1/3} \right) \right).$$

$$\text{(b)} \sup_{x \in \mathbb{R}} \left| P\left\{ I_{n,T} \left(-\frac{2\theta}{T} \right)^{1/2} (\hat{\theta}_{n,T,3} - \theta) \le x \right\} - \Phi(x) \right| = O\left(\max\left(T^{-1/2}, \left(\frac{T^3}{n^2} \right)^{1/3} \right) \right).$$

Proof (a) From (4.2), we have using Theorem 5.3 (c)

$$I_{n,T} \left(-\frac{2\theta}{T} \right)^{1/2} (\tilde{\theta}_{n,T,3} - \theta) = \left(-\frac{2\theta}{T} \right)^{1/2} Y_T + \theta \left(-\frac{2\theta}{T} \right)^{1/2} (I_T - I_{n,T}).$$

Hence, by Lemma 1.3, Lemma 1.5 and using Theorem 5.3 (c)

$$\begin{aligned}
&\sup_{x \in \mathbb{R}} \left| P\left\{ I_{n,T} \left(-\frac{2\theta}{T} \right)^{1/2} (\tilde{\theta}_{n,T,3} - \theta) \le x \right\} - \Phi(x) \right| \\
=\ & \sup_{x \in \mathbb{R}} \left| P\left\{ \left(-\frac{2\theta}{T} \right)^{1/2} Y_T + \theta \left(-\frac{2\theta}{T} \right)^{1/2} (I_T - I_{n,T}) \le x \right\} - \Phi(x) \right| \\
\le\ & \sup_{x \in R} \left| P\left\{ \left(-\frac{2\theta}{T} \right)^{1/2} Y_T \le x \right\} - \Phi(x) \right| + P\left\{ \left| \theta \left(-\frac{2\theta}{T} \right)^{1/2} (I_T - I_{n,T}) \right| > \epsilon \right\} + \epsilon \\
\le\ & CT^{-1/2} + C \frac{E|I_T - I_{n,T}|^2}{T\epsilon^2} + \epsilon \\
\le\ & CT^{-1/2} + C \frac{T^3}{n^2 \epsilon^2} + \epsilon.
\end{aligned}$$

Choosing $\epsilon = (\frac{T^3}{n^2})^{1/3}$, the theorem follows. □

(b) From (4.2), using Theorem 5.4 (c), we have

$$I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta) = \left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T}).$$

Hence, by Lemma 1.3, Lemma 1.5 and using Theorem 5.4 (c)

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\hat{\theta}_{n,T,3}-\theta)\le x\right\}-\Phi(x)\right| \\
=\ &\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T+\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\le x\right\}-\Phi(x)\right| \\
\le\ &\sup_{x\in R}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right|+P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T-I_{n,T})\right|>\epsilon\right\}+\epsilon \\
\le\ &CT^{-1/2}+C\frac{E|I_T-I_{n,T}|^2}{T\epsilon^2}+\epsilon \\
\le\ &CT^{-1/2}+C\frac{T^3}{n^2\epsilon^2}+\epsilon.
\end{aligned}$$

Choosing $\epsilon = (\frac{T^3}{n^2})^{1/3}$, the theorem follows. □

5. Stochastic Bounds for ABEs and AMAPEs

The following theorem gives bound on the error of approximation of the continuous BE by the discrete ABEs.

Theorem 5.1

$$(a)\quad |\tilde{\theta}_{n,T,1}-\tilde{\theta}_{T,1}| = O_P\left(\frac{T^2}{n}\right)^{1/2},$$

$$(b)\quad |\tilde{\theta}_{n,T,2}-\tilde{\theta}_{T,2}| = O_P\left(\frac{T^3}{n^2}\right)^{1/2},$$

$$(c)\quad |\tilde{\theta}_{n,T,3}-\tilde{\theta}_{T,3}| = O_P\left(\frac{T^4}{n^2}\right)^{1/2}.$$

Proof For $i = 1, 2, 3$, we have

$$\begin{aligned}
&\tilde{\theta}_{n,T,i} - \tilde{\theta}_{T,i} \\
= &\int_\Theta \theta \Lambda_{n,T,i}(\theta|X_0^{n,T})d\theta - \int_\Theta \theta \Lambda_{T,i}(\theta|X_0^T)d\theta \\
= &\int_\Theta \theta \left[\Lambda_{n,T,i}(\theta|X_0^{n,T}) - \Lambda_{T,i}(\theta|X_0^T)\right] d\theta \\
= &\int_\Theta \theta \left[\frac{\lambda(\theta)L_{n,T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{n,T,i}(\theta)d\theta} - \frac{\lambda(\theta)L_{T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{T,i}(\theta)d\theta}\right] d\theta \\
= &\int_\Theta \theta\lambda(\theta) \left[\frac{L_{n,T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{n,T,i}(\theta)d\theta} - \frac{L_{T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{T,i}(\theta)d\theta}\right] d\theta.
\end{aligned}$$

We have

$$|L_{n,T,1}(\theta) - L_{T,1}(\theta)| = O\left(\frac{T^2}{n}\right)^{1/2},$$

$$|L_{n,T,2}(\theta) - L_{T,2}(\theta)| = O\left(\frac{T^3}{n^2}\right),$$

$$|L_{n,T,3}(\theta) - L_{T,3}(\theta)| = O\left(\frac{T^4}{n^2}\right)^{1/2},$$

and hence

$$\left|\int_\Theta \theta\lambda(\theta)L_{n,T,1}(\theta)d\theta - \int_\Theta \theta\lambda(\theta)L_{T,1}(\theta)d\theta\right| = O\left(\frac{T^2}{n}\right)^{1/2},$$

$$\left|\int_\Theta \theta\lambda(\theta)L_{n,T,2}(\theta)d\theta - \int_\Theta \theta\lambda(\theta)L_{T,2}(\theta)d\theta\right| = O\left(\frac{T^3}{n^2}\right)^{1/2},$$

$$\left|\int_\Theta \theta\lambda(\theta)L_{n,T,3}(\theta)d\theta - \int_\Theta \theta\lambda(\theta)L_{T,3}(\theta)d\theta\right| = O\left(\frac{T^4}{n^2}\right)^{1/2}.$$

An application of Lemma 1.2 gives the result. □

The following theorem gives bound on the error of approximation of the continuous MAPE by the discrete AMAPEs.

Theorem 5.2

$$(a) \quad |\hat{\theta}_{n,T,1} - \hat{\theta}_{T,1}| = O_P\left(\frac{T^2}{n}\right)^{1/2},$$

$$(b)\quad |\hat{\theta}_{n,T,2} - \hat{\theta}_{T,2}| = O_P\left(\frac{T^3}{n^2}\right)^{1/2},$$

$$(c)\quad |\hat{\theta}_{n,T,3} - \hat{\theta}_{T,3}| = O_P\left(\frac{T^4}{n^2}\right)^{1/2}.$$

Proof. For $i = 1, 2, 3$, we have

$$\begin{aligned}
& \hat{\theta}_{n,T,i} - \hat{\theta}_{T,i} \\
= \; & \arg\max_{\theta\in\Theta} \Lambda_{n,T,i}(\theta|X_0^{n,T}) - \arg\max_{\theta\in\Theta} \Lambda_{T,i}(\theta|X_0^T) \\
= \; & \arg\max_{\theta\in\Theta} \left[\Lambda_{n,T,i}(\theta|X_0^{n,T}) - \Lambda_{T,i}(\theta|X_0^T)\right] \\
= \; & \arg\max_{\theta\in\Theta} \left[\frac{\lambda(\theta)L_{n,T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{n,T,i}(\theta)d\theta} - \frac{\lambda(\theta)L_{T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{T,i}(\theta)d\theta}\right] \\
= \; & \arg\max_{\theta\in\Theta} \lambda(\theta)\left[\frac{L_{n,T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{n,T,i}(\theta)d\theta} - \frac{L_{T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{T,i}(\theta)d\theta}\right].
\end{aligned}$$

We have

$$|L_{n,T,1}(\theta) - L_{T,1}(\theta)| = O\left(\frac{T^2}{n}\right)^{1/2},$$

$$|L_{n,T,2}(\theta) - L_{T,2}(\theta)| = O\left(\frac{T^3}{n^2}\right),$$

$$|L_{n,T,3}(\theta) - L_{T,3}(\theta)| = O\left(\frac{T^4}{n^2}\right)^{1/2}$$

and hence

$$\left|\int_\Theta \lambda(\theta)L_{n,T,1}(\theta)d\theta - \int_\Theta \lambda(\theta)L_{T,1}(\theta)d\theta\right| = O\left(\frac{T^2}{n}\right)^{1/2},$$

$$\left|\int_\Theta \lambda(\theta)L_{n,T,2}(\theta)d\theta - \int_\Theta \lambda(\theta)L_{T,2}(\theta)d\theta\right| = O\left(\frac{T^3}{n^2}\right)^{1/2},$$

$$\left|\int_\Theta \lambda(\theta)L_{n,T,3}(\theta)d\theta - \int_\Theta \lambda(\theta)L_{T,3}(\theta)d\theta\right| = O\left(\frac{T^4}{n^2}\right)^{1/2}.$$

An application of Lemma 1.2 gives the result. □

Theorem 5.3

$$a)\quad \sqrt{T}(\tilde{\theta}_{n,T,1}-\theta)=\sqrt{T}(\theta_{n,T,1}-\theta)+O_P\left(\frac{T^3}{n}\right)^{1/2}+o_P(1).$$

$$b)\quad \sqrt{T}(\tilde{\theta}_{n,T,2}-\theta)=\sqrt{T}(\theta_{n,T,2}-\theta)+O_P\left(\frac{T^4}{n}\right)^{1/2}+o_P(1).$$

$$c)\quad \sqrt{T}(\tilde{\theta}_{n,T,3}-\theta)=\sqrt{T}(\theta_{n,T,3}-\theta)+O_P\left(\frac{T^5}{n}\right)^{1/2}+o_P(1).$$

Proof: a) Using Theorem 5.1 (a), [13] and [15], we have

$$\begin{aligned}
&\sqrt{T}(\tilde{\theta}_{n,T,1}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,1}-\theta_{n,T,1})+\sqrt{T}(\theta_{n,T,1}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,1}-\tilde{\theta}_{T,1})+\sqrt{T}(\tilde{\theta}_{T,1}-\theta_{n,T,1})+\sqrt{T}(\theta_{n,T,1}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,1}-\tilde{\theta}_{T,1})+\sqrt{T}(\tilde{\theta}_{T,1}-\theta_{T,1})+\sqrt{T}(\theta_{T,1}-\theta_{n,T,1})+\sqrt{T}(\theta_{n,T,1}-\theta)\\
&=O_P\left(\frac{T^3}{n}\right)^{1/2}+o_P(1)+O_P\left(\frac{T^3}{n}\right)^{1/2}+\sqrt{T}(\theta_{n,T,1}-\theta).
\end{aligned}$$

b) Using Theorem 5.1 (b), [13] and [15] we have

$$\begin{aligned}
&\sqrt{T}(\tilde{\theta}_{n,T,2}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,2}-\theta_{n,T,2})+\sqrt{T}(\theta_{n,T,2}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,2}-\tilde{\theta}_{T,2})+\sqrt{T}(\tilde{\theta}_{T,2}-\theta_{n,T,2})+\sqrt{T}(\theta_{n,T,2}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,2}-\tilde{\theta}_{T,2})+\sqrt{T}(\tilde{\theta}_{T,2}-\theta_{T,2})+\sqrt{T}(\theta_{T,2}-\theta_{n,T,2})+\sqrt{T}(\theta_{n,T,2}-\theta)\\
&=O_P\left(\frac{T^4}{n}\right)^{1/2}+o_P(1)+O_P\left(\frac{T^4}{n}\right)^{1/2}+\sqrt{T}(\theta_{n,T,2}-\theta).
\end{aligned}$$

c) Using Theorem 5.1 (c), [13] and [16], we have

$$\begin{aligned}
&\sqrt{T}(\tilde{\theta}_{n,T,3}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,3}-\theta_{n,T,3})+\sqrt{T}(\theta_{n,T,3}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,3}-\tilde{\theta}_{T,3})+\sqrt{T}(\tilde{\theta}_{T,3}-\theta_{n,T,3})+\sqrt{T}(\theta_{n,T,3}-\theta)\\
&=\sqrt{T}(\tilde{\theta}_{n,T,3}-\tilde{\theta}_{T,3})+\sqrt{T}(\tilde{\theta}_{T,3}-\theta_{T,3})+\sqrt{T}(\theta_{T,3}-\theta_{n,T,3})+\sqrt{T}(\theta_{n,T,3}-\theta)\\
&=O_P\left(\frac{T^5}{n}\right)^{1/2}+o_P(1)+O_P\left(\frac{T^5}{n}\right)^{1/2}+\sqrt{T}(\theta_{n,T,3}-\theta).
\end{aligned}$$

Theorem 5.4

$$a)\quad \sqrt{T}(\hat{\theta}_{n,T,1}-\theta)=\sqrt{T}(\theta_{n,T,1}-\theta)+O_P\left(\frac{T^3}{n}\right)^{1/2}+o_P(1).$$

$$b)\quad \sqrt{T}(\hat{\theta}_{n,T,2}-\theta)=\sqrt{T}(\theta_{n,T,2}-\theta)+O_P\left(\frac{T^4}{n}\right)^{1/2}+o_P(1).$$

$$c)\quad \sqrt{T}(\hat{\theta}_{n,T,3}-\theta)=\sqrt{T}(\theta_{n,T,3}-\theta)+O_P\left(\frac{T^5}{n}\right)^{1/2}+o_P(1).$$

Proof: a) Using Theorem 5.2 (a), [13] and [15], we have

$$\begin{aligned}
&\sqrt{T}(\hat{\theta}_{n,T,1}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,1}-\theta_{n,T,1})+\sqrt{T}(\theta_{n,T,1}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,1}-\hat{\theta}_{T,1})+\sqrt{T}(\hat{\theta}_{T,1}-\theta_{n,T,1})+\sqrt{T}(\theta_{n,T,1}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,1}-\hat{\theta}_{T,1})+\sqrt{T}(\hat{\theta}_{T,1}-\theta_{T,1})+\sqrt{T}(\theta_{T,1}-\theta_{n,T,1})+\sqrt{T}(\theta_{n,T,1}-\theta)\\
&=O_P\left(\frac{T^3}{n}\right)^{1/2}+o_P(1)+O_P\left(\frac{T^3}{n}\right)^{1/2}+\sqrt{T}(\theta_{n,T,1}-\theta).
\end{aligned}$$

b) Using Theorem 5.2 (b), [13] and [15], we have

$$\begin{aligned}
&\sqrt{T}(\hat{\theta}_{n,T,2}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,2}-\theta_{n,T,2})+\sqrt{T}(\theta_{n,T,2}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,2}-\hat{\theta}_{T,2})+\sqrt{T}(\hat{\theta}_{T,2}-\theta_{n,T,2})+\sqrt{T}(\theta_{n,T,2}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,2}-\hat{\theta}_{T,2})+\sqrt{T}(\hat{\theta}_{T,2}-\theta_{T,2})+\sqrt{T}(\theta_{T,2}-\theta_{n,T,2})+\sqrt{T}(\theta_{n,T,2}-\theta)\\
&=O_P\left(\frac{T^4}{n}\right)^{1/2}+o_P(1)+O_P\left(\frac{T^4}{n}\right)^{1/2}+\sqrt{T}(\theta_{n,T,2}-\theta).
\end{aligned}$$

c) Using Theorem 5.2 (c), [13] and [16], we have

$$\begin{aligned}
&\sqrt{T}(\hat{\theta}_{n,T,3}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,3}-\theta_{n,T,1})+\sqrt{T}(\theta_{n,T,1}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,3}-\hat{\theta}_{T,1})+\sqrt{T}(\hat{\theta}_{T,3}-\theta_{n,T,3})+\sqrt{T}(\theta_{n,T,3}-\theta)\\
&=\sqrt{T}(\hat{\theta}_{n,T,3}-\hat{\theta}_{T,3})+\sqrt{T}(\hat{\theta}_{T,3}-\theta_{T,3})+\sqrt{T}(\theta_{T,3}-\theta_{n,T,3})+\sqrt{T}(\theta_{n,T,3}-\theta)\\
&=O_P\left(\frac{T^5}{n}\right)^{1/2}+o_P(1)+O_P\left(\frac{T^5}{n}\right)^{1/2}+\sqrt{T}(\theta_{n,T,3}-\theta).
\end{aligned}$$

6. Rate of Convergence in the Bernstein-von Mises Theorem

Interalia we obtain the rate of convergence of the approximate posterior distribution to normal distribution.

Theorem 6.1

$$a)\quad \sup_{x\in\mathbb{R}} |\Lambda_{n,T,1}(\theta|X_0^{n,T}) - \Phi(x)| = O(\alpha_{n,T}).$$

$$b)\quad \sup_{x\in\mathbb{R}} |\Lambda_{n,T,2}(\theta|X_0^{n,T}) - \Phi(x)| = O(\beta_{n,T}).$$

$$c)\quad \sup_{x\in\mathbb{R}} |\Lambda_{n,T,3}(\theta|X_0^{n,T}) - \Phi(x)| = O(\gamma_{n,T}).$$

Proof: Recall that the approximate posterior densities are given by

$$\Lambda_{n,T,i}(\theta|X_0^{n,T}) := \frac{\lambda(\theta)L_{n,T,i}(\theta)}{\int_\Theta \lambda(\theta)L_{n,T,i}(\theta)d\theta}, \qquad i = 1, 2, 3.$$

Let $u := \sqrt{T}(\theta - \theta_0)$ be the rescaled parameter. The approximate posterior distribution admits the density

$$\Lambda_{n,T,i}(u|X_0^{n,T}) := \frac{\lambda(\theta_0 + T^{-1/2}u)L_{n,T,i}(u)}{\int_{\mathbb{R}} \lambda(\theta_0 + T^{-1/2}v)L_{n,T,i}(v)dv}, \qquad i = 1, 2, 3.$$

Note that

$$\begin{aligned} &\int \left| \frac{\lambda(\theta_0 + T^{-1/2}u)L_{n,T,i}(u)}{\int_{\mathbb{R}} \lambda(\theta_0 + T^{-1/2}v)L_{n,T,i}(v)dv} - \frac{\lambda(\theta_0)L_{T,i}(u)}{\int_{\mathbb{R}} \lambda(\theta_0)L_{T,i}(v)dv} \right| du \\ &\le 2\Big(\int_{\mathbb{R}} \lambda(\theta_0)L_{T,i}(v)dv\Big)^{-1} \delta_n \end{aligned}$$

where

$$\delta_n := \int \Big| \lambda(\theta_0 + T^{-1/2}v)L_{n,T,i}(v) - \lambda(\theta_0)L_{T,i}(v) \Big| dv \to^P 0$$

as $n \to \infty$ and $T/n \to 0$ and

$$\left(\int_{\mathbb{R}} \lambda(\theta_0)L_{T,i}(v)dv\right)^{-1} = \frac{1}{\lambda(\theta_0)}\left(\frac{1}{2}(2\theta_0)I_T\right) = O_P(1).$$

Now the result follows from the rate of convergence in the Bernstein-von Mises theorem for continuous posterior density obtained in [13] and an application of Lemma 1.1(a). We omit the details.

Concluding Remarks

(1) The Berry-Esseen bounds are uniform over compact subsets of the parameter space.

(2) It remains to investigate the nonuniform rates of convergence to normality which are more useful.

(3) Large deviations of the ABEs and AMAPEs remains to be investigated.

(4) Sequential Bayes method/particle filtering method for O-U estimation problem remains to be investigated.

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